Journal of Engineering Mathematics, Vol. 13, No. 4, October 1979

Printed in the Netherlands

On Vlasov's equation with mollified density for an electron gas in an exterior magnetic field

W. STRAMPP*)

Department of Mathematics, Gesamthochschule Paderborn, Paderborn, W.–Germany.

(Received May 14, 1979)

SUMMARY

In this paper we consider Vlasov's equation with an exterior magnetic field. By using the mollification of the Coulomb potential introduced by Batt, classical solutions are established.
Furthermore, we study the effect of the exterior magnetic field on the solution. Choosing axially symmetric initial conditions we obtain solutions describing the confinement of an electron gas in a cylinder of infinite axial extent.

1. Statement of the problem

Let a system of charged particles move under the influence of an exterior magnetic force and of Coulomb forces generated by the particles themselves. We describe the system by a distribution function f depending on the time, t, and on the location (x,v) in the phase space. At a particular time t an observer sees the quantity $\int_V f(t,x,v)dxdv$ of particles in the volume V in the phase space. If collisions between the particles are excluded the distribution f remains constant along the path of a particle in the phase space. This is the content of Vlasov's equation (or collisionless Boltzmann equation).
Now we consider a gas of electrons and in particular a gas column of infinite axial extent. We ask whether the column can be stabilised by a constant magnetic field parallel to the column axis. By means of linear perturbation analysis applied to Vlasov's equation this problem was treated in [1], for example. We use a cut-off condition for the Coulomb potential and deal with the full equation.
We shall treat the gas column in the following way. To a solution of Vlasov's equation we assign at each time the mean distance $r_M(t)$ of an electron from the axis. By using flow-invariance methods we show that there are solutions such that $r_M(t)$ has an upper bound independent of the time t. It turns out that a certain amount of the magnetic field strength is sufficient to assure the bound.
Denote by $f\colon I \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ the distribution function (I is an arbitrary time interval, $\mathbb{R}^3 \times \mathbb{R}^3$ is the phase space). With an arbitrary exterior magnetic field B Vlasov's equation takes form:

$$f_t + v\nabla_x f - \frac{e}{m}\left(Ef + \frac{v \times B}{c}\right)\nabla_v f = 0. \tag{1}$$

* Supported by Forschungsförderung des Landes Nordrhein-Westfalen.

(c is the speed of light, $-e$ the charge and m the mass of the electrons). In equation (1) the force,

$$(Ef)(t,x) = -\nabla_x (Pf)(t,x),$$

is the gradient of the Coulomb potential,

$$(Pf)(t,x) = 4\pi e \int_{\mathbb{R}^3} \frac{\rho_f(t,x')}{\|x-x'\|}\, dx',$$

which is produced by the local density of the gas,

$$\rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv.$$

In [2] the existence of weak solutions of (1) is established. For the purpose of this paper we need differentiable solutions. Thus we introduce into equation (1) the smoothed density

$$\bar\rho_f(t,x) = \int_{\mathbb{R}^3} \omega_\delta(x-x')\,\rho_f(t,x')\,dx',$$

using the mollifier

$$\omega_\delta(x) = \delta^{-3}\omega_1\left(\frac{x}{\delta}\right),$$

$$\omega_1(x) = \begin{cases} \bar c \exp\left(-\dfrac{\|x\|^2}{1-\|x\|^2}\right), & \|x\| \le 1 \\ 0, & \|x\| \ge 1 \end{cases}$$

$$\left(\delta \text{ is arbitrary}, \frac{1}{\bar c} = \int_{\|x\|\le 1} \exp\left(-\frac{\|x\|^2}{1-\|x\|^2}\right) dx\right).$$

In an equivalent way the mollification may be interpreted as follows: since a short calculation yields

$$\int_{\mathbb{R}^3} \frac{\rho_f(t,x')}{\|x-x'\|}\, dx' = 4\pi e \int_{\mathbb{R}^3} \sigma(\|x-x'\|)\rho_f(t,x')\,dx'$$

where

$$\sigma(r) = \begin{cases} \dfrac{1}{r}\,. & r \ge \delta \\ 4\pi\left(\dfrac{1}{r}\displaystyle\int_0^r \omega_\delta(r')r'^2\,dr' + \int_r^\delta \omega_\delta(r')r'\,dr'\right), & r \le \delta \end{cases}$$

we can say that we insert instead of the potential (Pf) the cut-off potential

$$(\bar{P}f)(t,x) = 4\pi e \int_{\mathbb{R}^3} \sigma(\|x - x'\|)\rho_f(t,x')dx' \tag{2}$$

into Vlasov's equation.
Thus, we consider the mollified equation:

$$f_t + v\nabla_x f - \frac{e}{m}\left(\bar{E}f + \frac{v \times B}{c}\right)\cdot \nabla_v f = 0,$$

$$(\bar{E}f)(t,x) = -\nabla_x(\bar{P}f)(t,x),$$

and we prescribe an initial distribution

$$f(0,x,v) = f_0(x,v).$$

In [3], [5], [6] and [7] the existence of solutions of the mollified equation without exterior magnetic field was proved.
In the first part of this paper (Sections 2 and 5) we state the existence of solutions of (3) and the conservation of the total energy of the system. Then we show that the solutions are axisymmetric if the data are axisymmetric and formulate the conservation of the canonical angular momentum in case of a constant magnetic field B parallel to the x_3-axis.
In the second part (Sections 6 and 7) we assume axisymmetry. By means of the conservation principles and a lemma concerning flow invariance proved in Section 6 we shall estimate the mean distance of an electron from the x_3-axis, $r_M(t) = \int_{\mathbb{R}^3} (x_1^2 + x_2^2)^{1/2} \rho_f(t,x)dx$. The result can be stated thus: if the initial distribution f_0 is chosen suitably then $r_M(t)$ has an upper bound. Theorem 3 assures that the bound depends only on the data f_0 and B while the length of the time interval does not enter into the bound.
Finally, we remark that the method used in the second part could be applied also to the non-mollified equation if it were possible to find differentiable solutions with bounded density. Such solutions are given for the non-mollified equation without exterior force and with gravitational instead of Coulomb forces in [4] in case of spherically symmetric initial distributions.

2. Two lemmas

We introduce the notations:

$$\|x\| = (\Sigma\, |x_i|^2)^{1/2}, \quad |x| = \max\, |x_i|, \quad x \in \mathbb{R}^n$$

$$|f|_0 = \sup_{x \in \mathbb{R}^n} |f(x)| \quad \text{for bounded functions } f\colon \mathbb{R}^n \to \mathbb{R}^m$$

$$\text{and } |f|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(x)|dx \quad \text{for functions } f \in L^1(\mathbb{R}^n).$$

From [9, p. 181] we have the following lemma concerning mollified functions:

Lemma 1

Let $g \in L^1(\mathbb{R}^3)$. Then there holds for $\bar{g}(x) = \int_{\mathbb{R}^3} \omega_\delta(x - x') g(x') dx'$:

a) $|\bar{g}|_0 \leq \delta_1 |g|_{L^1(\mathbb{R}^3)}, \quad \delta_1 = \delta^{-3} |\omega_1|_0,$

$|\nabla \bar{g}|_0 \leq \delta^{-4} |\nabla \omega_1|_0 \, |g|_{L^1(\mathbb{R}^3)};$

b) $|\bar{g}|_{L^1(\mathbb{R}^3)} \leqslant |g|_{L^1(\mathbb{R}^3)}.$

Due to the mollification we obtain suitable bounds for the cut-off potential

$(\tilde{P}g)(x) = \int_{\mathbb{R}^3} \frac{\bar{g}(x')}{\|x - x'\|} dx'$ and its first and second derivatives.

Lemma 2

Let $g \in L^1(\mathbb{R}^3)$. Then there holds for $(\tilde{P}g)(x) = \int_{\mathbb{R}^3} \frac{\bar{g}(x')}{\|x - x'\|} dx'$:

a) $|\tilde{P}g|_0 \leq \delta_2 |g|_{L^1(\mathbb{R}^3)}, \qquad \delta_2 = 1 + 2\pi\delta^{-3} |\omega_1|_0;$

b) $\left| \frac{\partial \tilde{P}g}{\partial x_i} \right|_0 \leq \delta_3 |g|_{L^1(\mathbb{R}^3)}, \qquad \delta_3 = 1 + 4\pi\delta^{-3} |\omega_1|_0;$

c) $\left| \frac{\partial^2 \tilde{P}g}{\partial x_k \partial x_i} \right|_0 \leq \delta_4 |g|_{L^1(\mathbb{R}^3)}, \quad \delta_4 = \frac{4\pi}{3} + 4 + 13 |\nabla\omega_1|_0 \delta^{-4}.$

Proof: (*a*) results from Lemma 1 and

$$|\tilde{P}g(x)| \leq \int_{\|x' - x\| \geq 1} |\bar{g}(x')| dx' + \int_{\|x - x'\| \leq 1} \|x' - x\|^{-1} |\bar{g}(x')| dx'$$
$$\leq |\bar{g}|_{L^1(\mathbb{R}^3)} + |\bar{g}|_0 \cdot 2\pi.$$

Similarly, (*b*) follows from

$$\frac{\partial \tilde{P}g(x)}{\partial x_i} = \int_{\mathbb{R}^3} \frac{x_i' - x_i}{\|x' - x\|^3} \bar{g}(x') dx'.$$

By [8, p. 86] we have for $k \neq i$:

$$\frac{\partial^2 \tilde{P}g(x)}{\partial x_k \partial x_i} = \int_{\|x'-x\|\leq 1} (\bar{g}(x') - \bar{g}(x)) \frac{3(x'_k - x_k)(x'_i - x_i)}{\|x' - x\|^5} dx'$$

$$+ \int_{\|x'-x\|\geq 1} \bar{g}(x') \frac{3(x'_k - x_k)(x'_i - x_i)}{\|x' - x\|^5} dx'.$$

Therefore, in the case $k \neq i$ we obtain (c) from:

$$\frac{\partial^2 \tilde{P}g(x)}{\partial x_k \partial x_i} \leq 3\sqrt{3}\, |\nabla \bar{g}|_0 \int_{\|x'-x\|\leq 1} \|x' - x\|^{-2} dx'$$

$$+ 3 \int_{\|x'-x\|\geq 1} \|\bar{g}(x')\| dx'.$$

Using the corresponding formula for the second derivatives of $\tilde{P}g$ in the case $k = i$ [8, p. 86] we can complete the proof of (c).

3. The solution method

In order to solve (3) we introduce the Banach space

$$C' = \{f: I \rightarrow L^1(\Gamma) | f \text{ continuous}\}; \Gamma = \mathbb{R}^3 \times \mathbb{R}^3$$

endowed with the norm

$$|f|' = \sup_{t \in I} |f(t,\cdot)|_{L^1(\Gamma)},$$

(see [7]). $I = [0,a]$, $a > 0$ is an arbitrary time interval.
For each $f \in C'$ and $t \in I$ the components $(\bar{E}f)^i(t,\cdot)$ of the force have continuous derivatives with respect to x. Lemmas 2b and 2c imply that $\bar{E}$ together with its spatial derivatives is continuous on $I \times \Gamma$.
Now assume that $f_0 \in C^1(\Gamma) \cap L^1(\Gamma)$ and $B \in C^1(\mathbb{R}^3)$, $|B|_0 < \infty$. For all $f \in G$,

$$G = \left\{ f \in C' \,|\, |f|' \leq |f_0|_{L^1(\Gamma)} \right\},$$

we have by Lemma 2c

$$\left| \frac{\partial}{\partial x_k} (\bar{E}f)^i(t,\cdot) \right| \leq 4\pi e\, \delta_4 |f_0|_{L^1(\Gamma)} = C_1. \tag{4}$$

Thus, the right-hand side of the characteristic system

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = -\frac{e}{m}\left((\bar{E}f)(t,x) + \frac{v \times B}{c}\right), \quad f \in G \tag{5}$$

increases at most linearly with $|(x,v)|$. Therefore the solutions of (5), the phase maps

$$L_f(t,\tau,x,v) = (X_f(t,\tau,x,v),\ \ V_f(t,\tau,x,v)),$$

$$L_f(\tau,\tau,x,v) = (x,v),$$

exist and are continuously differentiable in $I \times I \times \Gamma$.

From the theory of partial differential equations of the first order we know that $\tilde{f}(t,x,v) = f_0(L_f(0,t,x,v))$ satisfies the equation

$$\tilde{f}_t + v\nabla_x \tilde{f} - \frac{e}{m}\left(\bar{E}f + \frac{v \times B}{c}\right)\nabla_v \tilde{f} = 0.$$

Hence, each fixed point of the operator

$$(Tf)(t,x,v) = f_0(L_f(0,t,x,v)), \quad f \in G$$

is a solution of (3).

Since the right-hand side of (5) is divergenceless the phase maps are measure-preserving, see [10, p. 96]. We have thus

$$TG \subset G.$$

4. An existence theorem for (3)

The assumption of [3], [5] and [6] about the initial distribution f_0 which assert that the operator T contracts if we have no exterior magnetic field are still sufficient for a constant exterior magnetic field B. For non-constant B we need stronger assumptions since the directional field in (5) has no longer bounded derivatives with respect to x.

Thus we assume:

a) $B \in C^1(\mathbb{R}^3, \mathbb{R}^3)$, B and ∇B^i are bounded

$f_0 \in C^1(\Gamma)$, $f_0 \geq 0$ ($f_0 \not\equiv 0$);

b) for all $(x,v) \in \Gamma$ we have:

(V) $\quad |f_0(x,v) - f_0(x',v')| \leq \mu(x,v)|(x,v) - (x',v')|$

for all (x',v') with $|(x,v)-(x',v')| \leq 1$;

c) $v^2 f_0(x,v) \in L^1(\Gamma)$ and $\mu \in L^1(\Gamma)$

if B is constant,

$\exp(v^2) f_0(x,v) \in L^1(\Gamma)$ and $\exp(v^2)\,\mu\,(x,v) \in L^1(\Gamma)$

if B is non-constant.

We state the existence of solutions of Vlasov's equation in

Theorem 1

If the conditions (V) are satisfied there is exactly one solution $f \in C'(\Gamma) \cap C^1(I \times \Gamma)$ of Vlasov's equation (3) and the law of conservation of energy holds:

$$\frac{d}{dt}\left[\int_\Gamma m\, v^2\, f(t,x,v)dxdv - e \int_\Gamma (\bar{P}f)(t,x)\, f(t,x,v)dxdv\right] = 0.$$

Proof: By Lemma 2b we have for $f \in G$:

$$|(\bar{E}f)^i\,(t,\cdot)|_0 \leq 4\pi e\,\delta_3\,|f(t,\cdot)|_{L^1(\Gamma)}. \tag{6}$$

Thus, we obtain from the equations (5) $\left(\cdot = \frac{\partial}{\partial t}\right)$:

$$|\dot{V}^i_f(t,\tau,x,v)| \leq \frac{e}{m}\,c_2 + \frac{e}{mc}\,|B|_0 \sum_{i=1}^{3} |V^i_f(t,\tau,x,v)|$$

$$c_2 := 4\pi e\,\delta_3\,|f_0|_{L^1(\Gamma)},$$

and by Gronwall's lemma:

$$\sum_{i=1}^{3} |V^i_f(t,\tau,x,v)| \leq \left(\sum_{i=1}^{3} |V_i| + c_3\right) \exp\left(c_4\,|t-\tau|\right)$$

$$c_3 := 6a\,c_2\,\frac{e}{m}, \quad c_4 := 3\,\frac{e}{mc}\,|B|. \tag{7}$$

Since it follows by (4), (6) and (7):

$$|\dot{V}^i_{f_1}(t,\tau,x,v) - \dot{V}^i_{f_2}(t,\tau,x,v)| \leq \frac{e}{m}\,4\pi e\,\delta_3\,|f_1(t,\cdot) - f_2(t,\cdot)|_{L^1(\Gamma)}$$

$$+\frac{e}{m} c_1 \sum_{i=1}^{3} |X^i_{f_1}(t,\tau,x,v) - X^i_{f_2}(t,\tau,x,v)|$$

$$+2\frac{e}{mc}\left(\sum_{i=1}^{3} |V_i| + c_3\right) \exp(c_4 \cdot 2a)$$

$$\cdot \max_{1\leq 1\leq 3} |\nabla B^i|_0 \sum_{i=1}^{3} |X^i_{f_1}(t,\tau,x,v) - X^i_{f_2}(t,\tau,x,v)|$$

$$+\frac{e}{m} |B|_0 \sum_{i=1}^{3} |V^i_{f_1}(t,\tau,x,v) - V^i_{f_2}(t,\tau,x,v)|,$$

we can show similarly that

$$\sum_{i=1}^{3} |X^i_{f_1}(t,\tau,x,v) - X^i_{f_2}(t,\tau,x,v)| + |V^i_{f_1}(t,\tau,x,v) - V^i_{f_2}(t,\tau,x,v)|$$
$$\leq c_5 \exp(A(v)) \cdot \left| \int_\tau^t |f_1(t',\cdot) - f_2(t',\cdot)|_{L^1(\Gamma)} dt' \right| \tag{8}$$

with

$$A(v) = c_6 + c_7 \max_{1\leq i\leq 3} |\nabla B^i|_0 \sum_{i=1}^{3} |v_i|$$

and constants c_5, c_6, c_7 independent of f_1 and $f_2 \in G$. Now we introduce the function

$$h(v) = \exp(1 + v^2)^{1/2}, \quad v \in \mathbb{R}^3,$$

and claim for $f \in G$:

$$\frac{h(v)}{h(V_f(0,t,x,v))} \leq c_8 \tag{9}$$

where the constant c_8 is independent of f. To see this we write

$$\frac{h(v)}{h(V_f(0,t,x,v))} = \exp \int_0^t \frac{\partial}{\partial\tau} ln(h(V_f(\tau,t,x,v)))d\tau.$$

Using (5) we find

$$\left|\frac{\partial}{\partial\tau} ln(h(V_f(\tau,t,x,v)))\right| \leq \frac{e}{m} \sum_{i=1}^{3} |(\bar{E}f)^i(\tau,X_f(\tau,t,x,v))|.$$

Since $t \leq a$, (9) follows by (6).

Next, we show that there is a constant c_9 such that for all $f_1, f_2 \in G$ and all $t \in I$

$$|f_0(L_{f_1}(0,t,x,v)) - f_0(L_{f_2}(0,t,x,v))|_{L^1(\Gamma)} \leq c_9 \int_0^t |f_1(t',\cdot) - f_2(t',\cdot)|_{L^1(\Gamma)} dt' \quad (10)$$

holds. To prove (10) let us choose α such that $A(v) \leq \alpha(1+v^2)^{1/2}$ for all $v \in \mathbb{R}^3$. (If B is constant we can set $\alpha = 0$ in the following). For fixed $t \in I$ we divide the phase space Γ into:

$$\Gamma_1 = \{(x,v) \in \Gamma \,||\, L_{f_1}(0,t,x,v) - L_{f_2}(0,t,x,v)| \leq 1\}$$

and

$$\Gamma_2 = \{(x,v) \in \Gamma \,||\, L_{f_1}(0,t,x,v) - L_{f_2}(0,t,x,v)| \geq 1\}.$$

By assumption (V) we have for $(x,v) \in \Gamma_1$:

$$|f_0(L_{f_1}(0,t,x,v)) - f_0(L_{f_2}(0,t,x,v))| \leq$$

$$h(V_{f_1}(0,t,x,v))^\alpha \mu(L_{f_1}(0,t,x,v)) \left(\frac{h(v)}{h(V_{f_1}(0,t,x,v))}\right)^\alpha \frac{|L_{f_1}(0,t,x,v) - L_{f_2}(0,t,x,v)|}{h(v)^\alpha}$$

Thus, taking (8) and (9) into account we obtain

$$\int_{\Gamma_1} |f_0(L_{f_1}(0,t,x,v)) - f_0(L_{f_2}(0,t,x,v))| dxdv \quad (11)$$
$$\int_\Gamma h(v)^\alpha \mu(x,v) dxdv \cdot c_8^\alpha \cdot c_5 \cdot \int_0^t |f_1(t',\cdot) - f_2(t',\cdot)|_{L^1(\Gamma)} dt'$$

since the phase maps are measure-preserving. (The integral $\int_\Gamma h(v)^\alpha \mu(x,v) dxdv$ exists by assumption (V)). Similarly, we can show that

$$\int_{\Gamma_2} |f_0(0,t,x,v)) - f_0(L_{f_2}(0,t,x,v))| dxdv \quad (12)$$
$$\leq 2 \int_\Gamma h(v)^\alpha f_0(x,v) dxdv \cdot c_8^\alpha \cdot \int_0^t |f_1(t',\cdot) - f_2(t',\cdot)|_{L^2(\Gamma)} dt'.$$

Then (11) and (12) imply (10).
Let us introduce in C' the norm

$$|f|^* = \sup_{t \in I} \left\{ \exp(-c_{10} t) \, |f(t,\cdot)|_{L^1(\Gamma)} \right\},$$

$$c_{10} > \max\{1, c_9\}.$$

This norm is equivalent to the norm $|\ |'$ and we obtain from (10):

$$\exp(-c_{10}t)\,|(Tf_1)(t,\cdot)-(Tf_2)(t,\cdot)|_{L^1(\Gamma)}$$

$$\leq c_9\cdot\exp(-c_{10}t)\cdot\frac{\exp(c_{10}t)-1}{c_{10}}\,|f_1-f_2|^*.$$

Thus T is a contracting operator:

$$|T_{f_1}-T_{f_2}|^*\leq\frac{c_9}{c_{10}}\,|f_1-f_2|^*,$$

and we have shown the first part of Theorem 1. Using (2) we can take the proof of [5] to show the conservation of energy since the magnetic field does not contribute to the energy balance. (The integral

$$\int_\Gamma v^2 f(t,x,v)dxdv=\int_\Gamma V_f(t,0,x,v)f_0(x,v)dxdv$$

exists by (7) and by assumption (V); the integral

$$\int_\Gamma(\bar{P}f)(t,x)\,f(t,x,v)dxdv=\int_\Gamma(\bar{P}f)(t,X_f(0,t,x,v))f_0(x,v)dxdv$$

exists by Lemma 2a.)

5. Axisymmetric solutions of (3)

We now consider axisymmetric magnetic fields. At first, let us introduce the notations:

$$r=(x_1^2+x_2^2)^{1/2},\qquad u=(v_1^2+v_2^2)^{1/2},$$

$$s=x_1v_1+x_2v_2,\qquad j=x_1v_2-x_2v_1,$$

and if $f\in C'$ and $L_f=(X_f,V_f)$ is the solution of the characteristic equations (5):

$$R_f=((X_f^1)^2+(X_f^2)^2)^{1/2},\qquad U_f=((V_f^1)^2+(V_f^2)^2)^{1/2},$$

$$S_f=X_f^1V_f^1+X_f^2V_f^2,\qquad J_f=X_f^1V_f^2-X_f^2V_f^1.$$

Denote by S the group of all rotations D of $\mathbb{R}^3$ leaving x_3 invariant:

$$(Dx)_3=x_3\quad\text{for all }x\in\mathbb{R}^3,$$

and by SC the set of all $f \in C^1(\Gamma)$ which are axysymmetric in both of its arguments:

$$f(x,v) = f(Dx,Dv) \quad \text{for all } (x,v) \in \Gamma,\ D \in S.$$

In addition to (V) we assume:

$$(\mathrm{V}_s) \quad \begin{aligned} &D(B(x)) = B(D(x)) \quad \text{for all } D \in S,\ \ x \in \mathbb{R}^3, \\ &f_0 \in SC. \end{aligned}$$

Theorem 2

Suppose the assumptions (V) and (V_s) are satisfied. Then the solution $f \in C'(\Gamma) \cap C^1(I \times \Gamma)$ of (3) has axysymmetric densities $\rho_f(t,x) = \rho_f(t,r,x_3)$ and $\bar{\rho}_f(t,x) = \bar{\rho}_f(t,r,x_3)$. Furthermore, if $B = (0,0,B^0)$ (B^0 constant) the canonical angular momentum is preserved:

$$\frac{d}{dt}\left[\, m\, J_f(t,\tau,x,v) - \frac{eB^0}{2c}\, R_f(t,\tau,x,v)^2 \right] = 0.$$

Proof: For $f \in C'$ it is easy to show:

a) if $f(t,\cdot) \in SC$ for all $t \in I$ then $\rho_f(t,\cdot)$ and $\bar{\rho}_f(t,\cdot)$ is axisymmetric.
Suppose we have shown
b) from $\bar{\rho}_f(t,x) = \bar{\rho}_f(t,r,x_3)$ it follows that $(Tf)\,(t,\cdot) \in SC$ for all $t \in I$.
The first part of Theorem 2 is then obtained in the following way. The solution $f \in C'(\Gamma) \cap C^1(I \times \Gamma)$ of (3) is the limit of the sequence $f_{n+1} = Tf_n$ starting from f_0. By (*a*) and (*b*) we have $f_n(t,\cdot) \in SC$ for all n. By Lemma 1:

$$\begin{aligned} |\bar{\rho}_{fn}(t,\cdot) - \bar{\rho}_f(t,\cdot)|_0 &\le \delta_1 \cdot |\rho_{f_n}(t,\cdot) - \rho_f(t,\cdot)|_{L^1(\mathbb{R}^3)} \\ &= \delta_1 \cdot |f_n(t,\cdot) - f(t,\cdot)|_{L^1(\Gamma)}. \end{aligned}$$

Therefore $\bar{\rho}_f(t,\cdot)$ is axisymmetric and by $Tf = f$ and (*a*), $\rho_f(t,\cdot)$ is axisymmetric too.
To show (*b*) we consider first the cut-off potential. Since $\bar{\rho}_f(t,\cdot)$ is axisymmetric we obtain for all $D \in S$:

$$(\bar{P}f)\,(t,x) = 4\pi e \int_{\mathbb{R}^3} \frac{\bar{\rho}_f(t,Dx')}{\|Dx' - Dx\|}\, dx'$$

$$= 4\pi e \int_{\mathbb{R}^3} \frac{\bar{\rho}_f(t,x')}{\|x' - Dx\|}\, dx'.$$

Hence, $(\bar{P}f)(t,x) = (\bar{P}f)(t,r,x_3)$ and

$$(\bar{E}f)^i(t,x) = -\frac{x_i}{r}\frac{\partial}{\partial r}(\bar{P}f)(t,r,x_3), \quad i = 1,2. \tag{13}$$

From (13) it follows:

$$D((\bar{E}f)(t,x)) = (\bar{E}f)(t,Dx),$$

and with the symmetry of B we have:

$$D(v \times B(x)) = Dv \times B(Dx)$$

for all $D \in S$, $(x,v) \in \Gamma$. Therefore, we have

$$\frac{d}{dt} DX_f(t,\tau,x,v) = DV_f(t,\tau,x,v),$$

$$\frac{d}{dt} DV_f(t,\tau,x,v) = -\frac{e}{m}\left[(\bar{E}f)(t,DX_f(t,\tau,x,v)) + \frac{1}{c} DV_f(t,\tau,x,v) \times B(DX_f(t,\tau,x,v))\right].$$

This implies:

$$DX_f(t,\tau,x,v) = X_f(t,\tau,Dx,Dv),$$

$$DV_f(t,\tau,x,v) = V_f(t,\tau,Dx,DV).$$

Hence, we get by the symmetry of f_0:

$$(Tf)(t,x,v) = (Tf)(t,Dx,Dv).$$

Now, suppose $B = (0,0,B^\circ)$, B° constant. Using the cylinder symmetry of the Coulomb force (13) we confirm by differentiating the conservation of the canonical angular momentum.

6. A lemma concerning flow invariance

In this section we prepare the estimation of the mean value $\int_{\mathbb{R}^3} r\,\rho_f(t,x)dx$. By using flow-invariance arguments (see [11]) we will prove:

Lemma 3

Let $\alpha(t), \sigma(t) \in C^1(I), \alpha(t) \geq 0$ for all $t \in I = [0, a]$ (or $I = [0, \infty)$) and the following assumptions be satisfied:

a) $\dot{\alpha}(t) = \dot{\sigma}(t)$,

$\dot{\sigma}(t) = -h_1(t) + h_2(t)$,

with $h_1, h_2 \in C(I)$;

b) there exist constants $0 < d_1 \leq d_2, 0 < d_4 \leq d_3$ such that for all $t \in I$:

$$d_1 \ \alpha(t) \leq h_1(t) \leq d_2 \ \alpha(t),$$
$$d_4 \leq h_2(t) \leq d_3 \quad ;$$

c) with the constants d_1, d_2, d_3, d_4 from (*b*) and further constants $\bar{c} > 0, \tilde{c} > 0$ we define in the (α,σ)-plane two ellipses:

$$\sigma^2 + d_1\alpha^2 - 2d_3\alpha = \bar{c},$$
$$\sigma^2 + d_2\alpha^2 - 2d_4\alpha = \tilde{c},$$

where we assume that the two ellipses have the same vertex on the positive α-axis:

$$\frac{d_3}{d_1} + \left(\left(\frac{d_3}{d_1}\right)^2 + \frac{\bar{c}}{d_1}\right)^{1/2} = \frac{d_4}{d_2} + \left(\left(\frac{d_4}{d_2}\right)^2 + \frac{\tilde{c}}{d_2}\right)^{1/2},$$

and further that $(\alpha(0), \sigma(0)) \in G$,

$$G = \left\{ (\alpha,\sigma) \in \mathbb{R}^2, \ \alpha \geq 0 \ \middle| \ \begin{matrix} \sigma^2 \leq c - d_1\alpha^2 + 2d_3\alpha, & \sigma \geq 0 \\ \sigma^2 \leq c - d_2\alpha^2 + 2d_4\alpha, & \sigma < 0 \end{matrix} \right\}.$$

Then we have for all $t \in I$:

$$(\alpha(t), \ \sigma(t)) \in G, \ \text{ i.e.,}$$
$$\alpha(t) \leq \frac{d_3}{d_1} + \left(\left(\frac{d_3}{d_1}\right)^2 + \frac{\bar{c}}{d_1}\right)^{1/2}.$$

Proof: Denote by $a(t)$ the square of the distance between the point $\chi(t) = (\alpha(t), \sigma(t))$ and G. Since $\chi(0) \in G$ we have $a(0) = 0$. Suppose for some $t \in I$, $t > 0, a(t) > 0$. We want to show that this assumption implies

$$D_- a(t) \leq L \ a(t) \tag{14}$$

where D_- denotes a Dini-derivative (see [12, p. 551]) and L is a constant independent of t. Since $a(0) = 0$, we obtain from (14) with a theorem of [12, p. 60] $a(t) = 0$, as claimed.
Now, suppose $a(t) > 0$. Then, there exists a point $\bar{\chi} = (\alpha,\sigma)$ contained in

$$\epsilon 1) = \left\{ (\alpha,\sigma) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} \alpha \geq 0 \\ \sigma \geq 0 \end{matrix},\ \sigma^2 = \bar{c} - d_1\alpha^2 + 2d_3\alpha \right\}$$

or in

$$\epsilon 2) = \left\{ (\alpha,\sigma) \in R^2 \;\middle|\; \begin{matrix} \alpha \geq 0 \\ \sigma < 0 \end{matrix},\ \sigma^2 = \tilde{c} - d_2\alpha^2 + 2d_4\alpha \right\}$$

with the property that $a(t) = ||\chi(t) - \bar{\chi}||^2$.
To prove (14) we introduce the function

$$b(s) = ||\chi(t+s) - \bar{\chi}|| \quad \text{for small } s.$$

from the definitions of D_-, $a(t)$ and $b(s)$ the inequality:

$$D_- a(t) \leq b'(0)$$

follows. This was the framework of the proof which we have taken from a theorem of [11, p. 68]. Our task in the following is to estimate

$$b'(0) = 2\,(\chi(t) - \bar{\chi})\,\dot{\chi}(t).$$

Since $d_3 \geq 0$, $\epsilon 1)$ has one vertex contained in $\{\alpha \geq 0, \alpha > 0\}$ and since $d_4 \geq 0$, $\epsilon 2)$ has one vertex contained in $\{\alpha \geq 0,\ \sigma < 0\}$. Thus, the vector $\chi(t) - \bar{\chi}$ is up to a positive factor equal to the outward normal to G at $\bar{\chi}$.
Now we consider first the case $\sigma(t) \geq 0$. This implies $\bar{\sigma} \geq 0$. Thus, the vector $\chi(t) - \bar{\chi}$ is up to a positive factor equal to the vector

$$(d_1\bar{\alpha} - d_3, \bar{\sigma})$$

which is the outward normal to $\epsilon 1)$ at the point $\bar{\chi}$.
1.1) $0 \geq d_1 \cdot \alpha - h_1(t)$. Then the second equality of (a) and the second inequality of (b) imply:

$$(d_1\bar{\alpha} - d_3)\,\bar{\sigma} + \bar{\sigma}\,\dot{\sigma}(t) \leq \bar{\sigma}(d_1\bar{\alpha} - h_1(t)) \leq 0.$$

Thus, we have $(\chi(t) - \bar{\chi})\,(\bar{\sigma}, \dot{\sigma}(t)) \leq 0$ and by (a):

$$b'(0) \leq 2(\chi(t) - \bar{\chi})(\dot{\chi} - (\bar{\sigma}, \dot{\sigma}(t))$$

$$= 2(\chi(t) - \bar{\chi})(\sigma(t) - \bar{\sigma}, 0).$$

Hence,

$$b'(0) \leq 2\|\chi(t) - \chi\|^2.$$

1.2) $0 \leq d_1\bar{\alpha} - h_1(t)$. At first, the second inequality of (b) implies:

$$(d_1\bar{\alpha} - d_3)\bar{\sigma} + \bar{\sigma}(-d_1\bar{\alpha} + h_2(t)) \leq 0.$$

Thus, we have $(\chi(t) - \bar{\chi})(\bar{\sigma}, -d_1\bar{\alpha} + h_2(t)) \leq 0$ and by (a):

$$b'(0) \leq 2(\chi(t) - \bar{\chi})(\dot{\chi}(t) - (\bar{\sigma}, -d_1\bar{\alpha} + h_2(t))$$

$$= 2(\chi(t) - \bar{\chi})(\sigma(t) - \bar{\sigma}, -h_1(t) + d_1\bar{\alpha}).$$

Since, by assumption and by the first inequality of (b)

$$0 \leq d_1\bar{\alpha} - h_1(t) \leq d_1\bar{\alpha} - d_1\alpha(t),$$

we have:

$$b'(0) \leq 2 \max\{1, d_1\} \|\chi(t) - \bar{\chi}\|^2.$$

Now we consider the case $\sigma(t) \leq 0$. This implies $\bar{\sigma} \leq 0$. In this case the vector $\chi(t) - \bar{\chi}$ is up to a positive factor equal to

$$(d_2\bar{\alpha} - d_4, \bar{\sigma})$$

which is the outward normal to $\epsilon 2)$ at $(\bar{\alpha}, \bar{\sigma})$. Again, we distinguish between two subcases.

(2.1) $0 \geq d_2\bar{\alpha} - h_1(t)$. The second inequality of (b) implies:

$$(d_2\bar{\alpha} - d_4)\bar{\sigma} + \bar{\sigma}(-d_2\bar{\alpha} + h_2(t)) \leq 0.$$

Thus, by (a),

$$b'(0) \leq 2(\chi(t) - \bar{\chi})(\dot{\chi}(t) - (\bar{\sigma}, -d_2\bar{\alpha} + h_2(t))$$

$$= 2(\chi(t) - \bar{\chi})(\sigma(t) - \bar{\sigma}, -h_1(t) + d_2\bar{\alpha}).$$

Since, by assumption and by the first inequality of (b),

$$0 \leq h_1(t) - d_2\bar{\alpha} \leq d_2 \cdot \alpha(t) - d_2\bar{\alpha},$$

we have:

$$b'(0) \leq 2 \max \{1, d_2\} \, ||\chi(t) - \chi||^2 .$$

2.2) $0 \leq d_2\bar{\alpha} - h_1(t)$. Then, the second equality of (a) and the second inequality of (b) imply:

$$(d_2\bar{\alpha} - d_4)\bar{\sigma} + \bar{\sigma}\,\dot{\sigma}(t) \leq \bar{\sigma}\,(d_2\bar{\alpha} - h_1(t)) \leq 0.$$

Thus, we have by (a):

$$b'(0) \leq 2\,(\chi(t) - \bar{\chi})\,(\dot{\chi} - (\bar{\sigma},\dot{\sigma}(t))$$

$$= 2\,(\chi(t) - \bar{\chi})\,(\sigma(t) - \bar{\sigma},0).$$

Hence,

$$b'(0) \leq 2\,||\chi(t) - \bar{\chi}||^2 .$$

Finally, we obtain with a constant L:

$$b'(0) \leq L\,a(t),$$

since $||\chi(t) - \bar{\chi}||^2 = a(t)$. Thus, (14) and the lemma is shown.
We remark that Lemma 3 also applies to the differential equation:

$$\dot{\alpha} = \sigma,$$

$$\dot{\sigma} = h_1(t,\alpha) + h_2(t), \quad h_1 \in C(I \times \mathbb{R}), h_2 \in C(I),$$

where $d_1\alpha \leq h_1(t,\alpha) \leq d_2$ for all $(t,\alpha) \in I \times \mathbb{R}$ and $d_4 \leq h_2(t) \leq d_3$ for all $t \in I$. $(0 < d_1 \leq d_2$, $0 < d_4 \leq d_3)$. If we have a solution $(\alpha(t), \sigma(t))$ and $\alpha(t) \geq 0$ for all t then this solution remains bounded for all times by Lemma 3.

7. An estimate for the column radius

In this section we define appropriate functions $\alpha(t)$ and $\sigma(t)$ and show that the conditions of Lemma 3 are satisfied with these functions. The estimate given in Lemma 3 for $\alpha(t)$ will serve us to estimate the mean value $\int_{\mathbb{R}^3} r\,\rho_f(t,x)dx$.

Lemma 4

Suppose (V) and (V_s) are satisfied, $\int_\Gamma r^2 f_0(x,v)dxdv < \infty$, $B = (0,0,B^\circ)$ and B° constant.

Let $f \in C'(\Gamma) \cap C^1(J \times \Gamma)$ be the solution of (3). Then the conditions (a) of Lemma 3 are satisfied with the functions

$$\alpha(t) = \int_\Gamma R_f(t,0,x,v)^2 f_0(x,v)dxdv,$$

$$\sigma(t) = 2 \int_\Gamma S_f(t,0,x,v) f_0(x,v)dxdv,$$

$$h_1(t) = \int_\Gamma \left[\omega^2 R_f(t,0,x,v)^2 + \frac{2e}{m} \sum_{i=1}^{2} X_f^i(t,0,x,v)\,(\bar{E}f)^i\,(t,X_f(t,0,x,v))\right] f_0(x,v)dxdv,$$

$$h_2(t) = 2 \int_\Gamma u^2 f(t,x,v)dxdv + 2\omega \int_\Gamma \left(\omega \frac{r^2}{2} - j\right) f_0(x,v)dxdv,$$

where $\omega = \dfrac{eB^\circ}{mc}$.

Proof: The integral $\alpha(t) = \int_\Gamma R_f(t,0,x,v)^2 f_0(x,v)dxdv$ exists by assumption on f_0 and the estimate

$$|X_f^i(t,0,x,v)| \leq |x_i| + \left(\sum_{i=1}^{3} |v_i| + c_3\right) \int_0^a \exp(c_4 t)dt$$

which follows from (7). By similar arguments the existence of $\sigma(t)$ follows.
We obtain by differentiating:

$$\dot{\alpha}(t) = 2 \int_\Gamma S_f(t,x,v) f_0(x,v)dxdv,$$

$$\dot{\sigma}(t) = 2 \int_\Gamma U_f(t,0,x,v)^2 f_0(x,v)dxdv$$

$$- \frac{2e}{m} \int_\Gamma \sum_{i=1}^{2} X_f^i(t,0,x,v)\,(\bar{E}f)^i\,(t,X_f(t,0,x,v) f_0(x,v)dxdv$$

$$- 2\,\frac{eB^\circ}{mc} \int_\Gamma J_f(t,0,x,v) f_0(x,v)dxdv.$$

We are justified to differentiate under the integral sign since for all functions under the integrals on the right-hand side we have integrable majorants.
Making use of the conservation of the angular momenttum we obtain

$$-2\,\frac{eB^\circ}{mc} \int_\Gamma J_f(t,0,x,v) f_0(x,v)dxdv = 2\omega \int_\Gamma \left(\omega \frac{r^2}{2} - j\right) f_0(x,v)dxdv$$

$$- \int_\Gamma \omega^2 R_f(t,0,x,v)^2 f_0(x,v)dxdv$$

and this gives Lemma 4.

The following Lemma relies essentially on the symmetry of the Coulomb force.

Lemma 5

Let the assumptions of Lemma 4 be satisfied. If we set:

$$d_1 = \omega^2 - \frac{e}{m}((4\pi)^2 e\delta_1 + \delta_4)\,|f_0|_{L^1(\Gamma)},$$

$$d_2 = \omega^2 + \frac{e}{m}((4\pi)^2 e\delta_1 + \delta_4)\,|f_0|_{L^1(\Gamma)},$$

$$d_3 = 2\int_\Gamma v^2 f_0(x,v)dxdv - 2\,\frac{e}{m}\int_\Gamma (\overline{P}f_0)(x,v) f_0(x,v)dxdv$$

$$+ 2\,\frac{e}{m}\,\delta_2\left(|f_0|_{L^1(\Gamma)}\right)^2 + d_4,$$

$$d_4 = 2\,\frac{\omega}{m}\int_\Gamma \left(\omega m\,\frac{r^2}{2} - mj\right) f_0(x,v)dxdv$$

with the constants δ_1 of Lemma 1a, δ_2 of Lemma 2a and δ_4 of Lemma 2c and if we assume

$$d_1 > 0,\quad d_4 > 0,$$

then the conditions (b) of Lemma 3 are satisfied.
$(\delta_1 |f_0]_{L^1(\Gamma)}$ is a bound for the density $\overline{\rho}_f(t,x)$ of the electron gas. The cut-off potential $(\overline{P}f)(t,x)$ is bounded by $\delta_2 \left(|f_0|_{L^1(\Gamma)}\right)^2$.
This follows from Lemma 1 and 2 and furthermore, by Lemma 2, $\delta_4 |f_0|_{L^1(\Gamma)}$ is a bound for the second order spatial derivatives of $(\overline{P}f)(t,x)$. The further terms appearing in d_1 to d_4 refer to the total energy and to the canonical angular momentum of the gas).

Proof of Lemma 3: By using the symmetry of the force (14) we obtain:

$$h_1(t) = \int_\Gamma \left[\omega^2 - \frac{2e}{m}\frac{1}{R_f(t,0,x,v)}\frac{\partial \overline{P}_f}{\partial r}(t, R_f(t,0,x,v), X_f^3(t,0,x,v)\right] \cdot$$

$$R_f(t,0,x,v)^2 f_0(x,v)dxdv.$$

Since $\overline{P}f$ satisfies Poisson's equation,

$$\Delta_x(\overline{P}f)(t,x) = -(4\pi)^2\, e\overline{\rho}(t,x),$$

and in cylindrical coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r}\,(\overline{P}f)(t,r,x_3) + \frac{\partial^2}{\partial x_3^2}\,(\overline{P}f)(t,r,x_3) = -(4\pi)^2\, e\overline{\rho}(t,r,x_3),$$

we have by Lemma 1a and 2c:

$$\left|\frac{1}{r}\frac{\partial\overline{P}f}{\partial r}\right|_0 \leq \tfrac{1}{2}((4\pi)^2\, e\delta_1 + \delta_4)\,|f_0|_{L^1(\Gamma)}.$$

Thus:

$$d_1\alpha(t) \leq h_1(t) \leq d_2\alpha(t).$$

With the help of the conservation of the energy and Lemma 2a we have

$$\int_\Gamma u^2 f(t,x,v)dxdv \leq \int_\Gamma v^2 f(t,x,v)dxdv - \frac{e}{m}\int_\Gamma (\overline{P}f)(t,x)\,f(t,x,v)dxdv$$

$$+ \frac{e}{m}\int_\Gamma (\overline{P}f)(t,X_f(t,0,x,v))\,f_0(x,v)dxdv$$

$$\leq \int_\Gamma V^2 f_0(x,v)dxdv - \frac{e}{m}\int_\Gamma (\overline{P}f)(x)\,f_0(x,v)dxdv$$

$$+ \frac{e}{m}\,\delta_2\,\left(|f_0|_{L^1(\Gamma)}\right)^2.$$

Thus:

$$d_4 \leq h_2(t) \leq d_3.$$

Lemma 6

If the assumptions of Lemma 5 are satisfied then there exist constants $\overline{c}_i, \widetilde{c}_i$ $(i = 1,2)$ such that the condition (c) of Lemma 3 is satisfied with $\overline{c}_1, \widetilde{c}_1$ if $\sigma(0) \geq 0$ and with $\overline{c}_2, \widetilde{c}_2$ if $\sigma(0) < 0$.

Proof: If $\sigma(0) \geq 0$ we introduce the function

$$g(c) = d_2\left(\frac{d_3}{d_1} + \left(\left(\frac{d_3}{d_1}^2\right)^2 + \frac{c}{d_1}\right)^{1/2}\right) - 2\,d_4$$

and the constant

$$c_0 = \sigma(0)^2 + d_1\alpha(0)^2 - 2d_3\alpha(0).$$

Then, we choose $\bar{c}_1 > 0$ with $\bar{c}_1 \geq c_0$ and set

$$\tilde{c}_1 = g(\bar{c}_1)\left(\frac{d_3}{d_1} + \left(\left(\frac{d_3}{d_1}\right)^2 + \frac{\bar{c}_1}{d_1}\right)^{1/2}\right).$$

Hence, by definition, $\nu = \dfrac{d_3}{d_1} + \left(\left(\dfrac{d_3}{d_1}\right)^2 + \dfrac{\tilde{c}_1}{d_1}\right)^{1/2}$ solves:

$$\tilde{c}_1 - d_2\nu^2 + 2d_4\nu = 0 \tag{15}$$

Since (15) has only one positive solution, we have

$$\frac{d_3}{d_1} + \left(\left(\frac{d_3}{d_1}\right)^2 + \frac{\bar{c}_1}{d_1}\right)^{1/2} = \frac{d_4}{d_2} + \left(\left(\frac{d_4}{d_2}\right)^2 + \frac{\tilde{c}_1}{d_2}\right)^{1/2}$$

Obviously, $(\alpha(0), \sigma(0)) \in G$.
If $\sigma(0) < 0$ we introduce the function

$$g(c) = d_1\left(\frac{d_4}{d_2} + \left(\left(\frac{d_4}{d_2}\right)^2 + \frac{c}{d_2}\right)^{1/2}\right) - 2d_3$$

and the constant

$$c_0 = \sigma(0)^2 + d_2\alpha(0)^2 - 2d_4\alpha(0).$$

Then we choose a $\tilde{c}_2 > 0$ such that $\tilde{c}_2 \geq c_0$ and $g(\tilde{c}_2) > 0$ and we set:

$$\bar{c}_2 = g(\tilde{c}_2)\left(\frac{d_4}{d_2} + \left(\left(\frac{d_4}{d_2}\right)^2 + \frac{\tilde{c}_2}{d_2}\right)^{1/2}\right).$$

As above it is easy to see that with $\bar{c}_2$ and $\tilde{c}_2$ the conditions of Lemma 3 are satisfied.
Now, we are able to estimate the mean distance

$$r_M(t) = \int_{\mathbb{R}^3} r\rho_f(t,x)dx$$

of the electrons from the x_3-axis. Since

$$r_M(t) \leq |f_0|_{L^1(\Gamma)} + \int_\Gamma R_f(t,0,x,v)^2 f_0(x,v)dxdv.$$

we get from foregoing lemmas that in the mean the particle orbit is contained in a column of infinite axial extent.

Theorem 3

Under the assumptions of Lemma 5 the following holds:

$$r_M(t) \leq |f_0|_{L^1(\Gamma)} + \frac{d_3}{d_1} + \left(\left(\frac{d_3}{d_1}\right)^2 + \frac{\bar{c}_1}{d_1}\right)^{1/2},$$

if $\int_\Gamma s f_0(x,v) dx dv \geq 0$, and

$$r_M(t) \leq |f_0|_{L^1(\Gamma)} + \frac{d_3}{d_1} + \left(\left(\frac{d_3}{d_1}\right)^2 + \frac{\bar{c}_2}{d_1}\right)^{1/2},$$

if $\int_\Gamma s f_0(x,v) dx dv < 0$.

REFERENCES

[1] R.C., Davidson, *Theory of nonneutral plasmas.* W. A. Benjamin, Inc. Reading, Massachussets (1974).
[2] A. A. Arsenjev, Global existence of a weak solution of Vlasov's system of equations, *Zh. vȳchisl. Mat. mat. Fiz.* 15, 1 (1975) 136-147.
[3] J. Batt, Ein Existenzbeweis für die Vlasov-Gleichung der Stellardybamik bei gemittelter Dichte, *Arch. Rat. Mech. Anal.* 13 (1963) 296-308.
[4] J. Batt, Global symmetric solutions of the initial value problem of stellardynamics. *Journal of Diff. Equations* 25 (1977) 342-364.
[5] J. Batt, E. Horst, Global existence and conservation. Laws for the Vlasov equation. Unpublished.
[6] R. Rautmann, Lösung einer Anfangswertaufgabe für substantielle Differentialgleichungen. *Lecture Notes in Mathematics*, Vol. 315 (1974), Springer, Berlin. Heidelberg, New York, 85-108.
[7] G. Hannoschöck, R. Rautmann, Schwache Lösungen und Fehlerabschätzungen für die Vlasov Gleichung bei gemittelter Dichte, *ZAMM* 56 (1976) T 254-T 256.
[8] N. M. Günter, *Die Potentialtheorie und ihre Anwendung und Grundaufgaben der Mathematischen Physik.* Teubner. Leibzig (1957).
[9] W. I. Smirnov, *Lehrgang des höheren Mathematik*, Teil V. VEB deutscher Verlag der Wissenschaften, Berlin (1962).
[10] P. Hartman, *Ordinary Differential Equations.* John Wiley & Sons Inc., New York, London, Sidney, (1964).
[11] K. Deimling, *Ordinary differential equations in Banach spaces.* Lecture Notes in Mathematics, Vol. 596 (1977).
[12] W. Walter, *Differential und Integral Ungleichungen.* Springer, Berlin, Heidelberg, New York (1964).